A safe & computable approximation to Kolmogorov complexity

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preliminaries: Kolmogorov complexity

$$T_i(p)$$

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$$K(x) = \min_p \{ p : U(p) = x \}$$

preliminaries: Kolmogorov complexity

$$U(\bar{\imath}p) = T_i(p)$$
$$K(x) = \min_p \{p : U(p) = x\}$$

- U is a formalisation of the notion of a description
- K is invariant to the choice of U
  (up to a constant)

motivation

"Kolmogorov is not computable, it's only of theoretical use"

No, approximations are usually correct

preliminaries: Probabilities & codes

- $L(x)$: (prefix) code length function
- $p(x)$: probability (semi) measure

$$-\log p(x) = L(x)$$

step 1: computable probabilities

From TMs to probabilities:

$$T(p) = x$$

$$p_T(x) = \sum_{p:T(p) = x} 2^{-|p|}$$

$$m(x) = p_U(x)$$

equivalent to the lower semicomputable semimeasures

step 2: model classes

A model class C is an effectively enumerable subset of all Turing machines.

$$U^C(\bar{\imath}p) = T_i(p)$$

$$K^C(x) = \min_p \{p : U^C(p) = x\}$$

$$m^C(x) = \sum_{p:U^C(p)=x} 2^{-|p|}$$

step 3: safe approximation

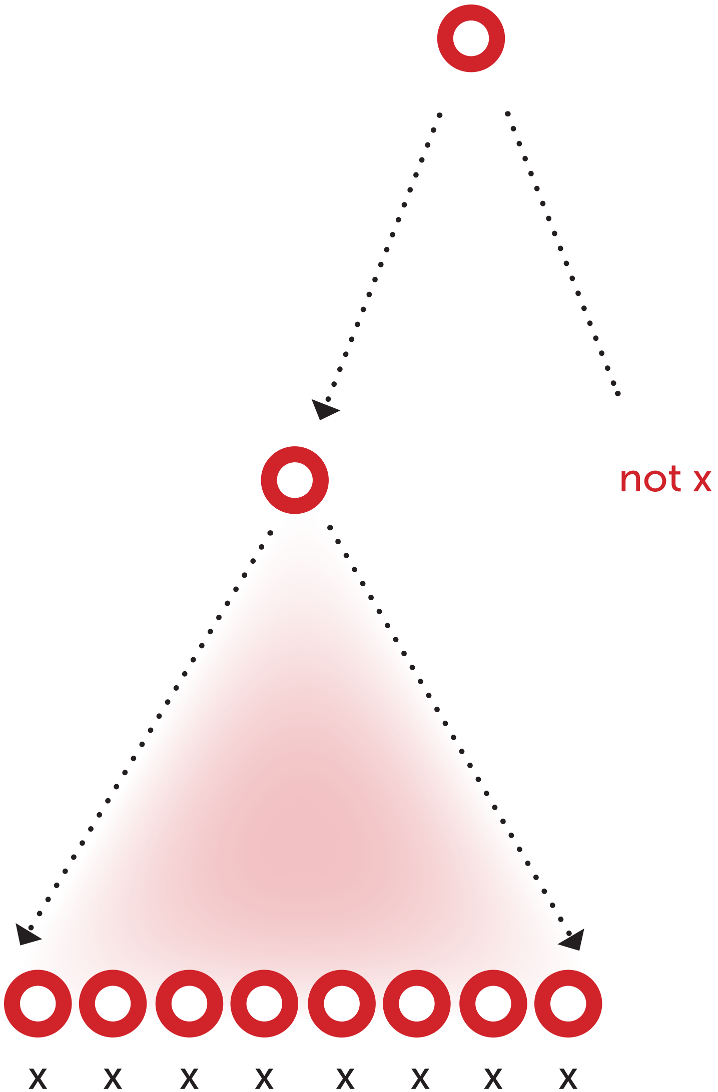
- $L(x)$: approximating code-length function
- $L(x)$ is safe *against* p when

$$p\big(L(x) - K(x) \geqslant k\big) \leqslant cb^{-k}$$

for some c and $b > 1$

Is K^C **safe against** $p \in C$**?**

no.



not x
x
x
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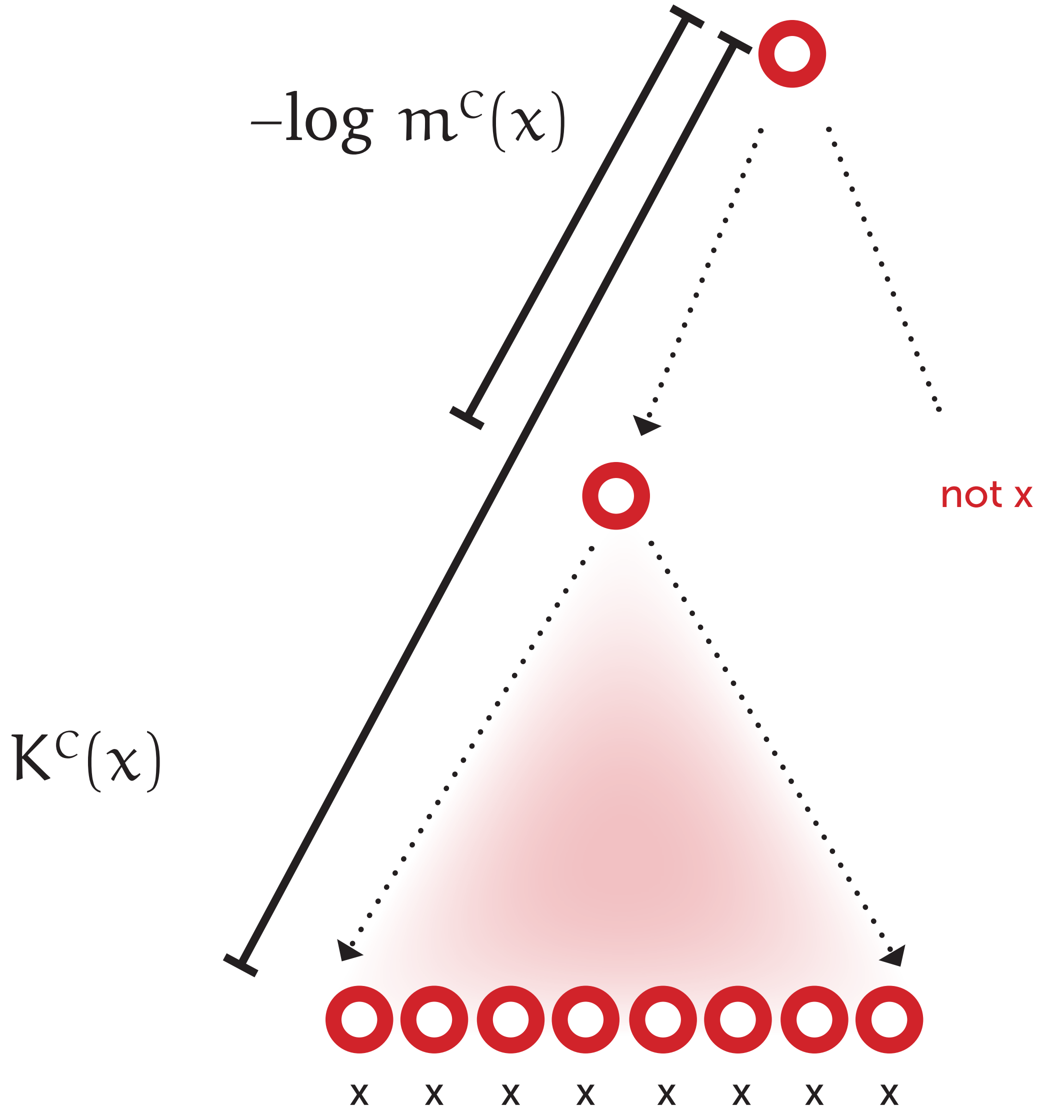

$-\log m^C(x)$
$K^C(x)$
not x
x x x x x x x x

Is $-\log m^{C}$ **safe against** $p \in C$**?**

yes.

-log $\mathbf{m}^C$ is safe against $\mathbf{m}^C$

$$\mathbf{m}^C\left(-\log \mathbf{m}^C(x) - K(x) \geq k\right)$$

-log m^{C} is safe against m^{C}

$$\mathrm{m}^{C}\left(-\log \mathrm{m}^{C}(x) - K(x) \geqslant k\right)$$

$$= \mathrm{m}^{C}\left(\mathrm{m}^{C}(x) \leqslant 2^{-k} 2^{-K(x)}\right)$$

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$$= \sum_{x : m^C(x) \leq 2^{-k} 2^{-K(x)}} m^C(x)$$

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$$\leqslant \sum 2^{-k}2^{-K(x)}$$

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$$\leqslant \sum 2^{-k} 2^{-K(x)}$$

$$= 2^{-k} \sum 2^{-K(x)} \leqslant 2^{-k}$$

-log m^C is safe against members of C

$$m^C(\cdot) = \sum c_q p_q(\cdot) \geqslant c_q p_q(\cdot)$$

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$$c_q p_q\left(-\log m^C(x) - K(x) \geqslant k\right)$$

$$\leqslant m^C\left(-\log m^C(x) - K(x) \geqslant k\right)$$

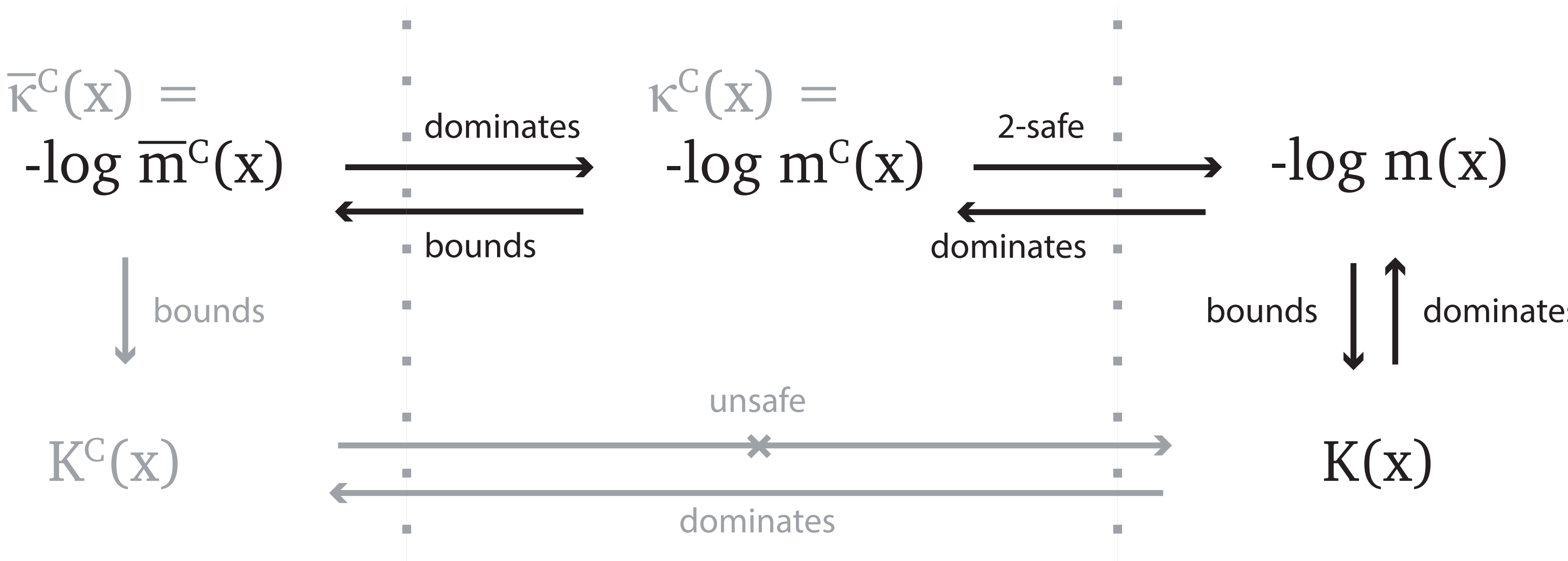
$$\leqslant 2^{-k}$$

can we compute m^C?

- We can if it's upper and lower semicomputable
- lower: dovetail all programs for U^C
- upper: dovetail until

$$(1-s)/s_x \leqslant 2^c - 1$$

- If C is complete, this algorithm is computable



$\overline{\kappa}^C(x) =$
$-\log \overline{m}^C(x)$
dominates
bounds
$\kappa^C(x) =$
$-\log m^C(x)$
2-safe
dominates
$-\log m(x)$
bounds
bounds
dominates
unsafe
$K^C(x)$
dominates
$K(x)$

What does this buy us?

- bridge between the practical and the platonic
- Bayesian $\leftrightarrow$ MDL $\leftrightarrow$ Algorithmic
- corollary: K^t
- Additional results: ID, NID

Questions?